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Calculational and Conceptual Orientations in Teaching Mathematics

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How mathematics curriculum reform is implemented in the classroom depends largely on teachers' images of the mathematics they are teaching (Bauersfeld 1980; Cooney 1985; Thompson 1984). From our close collaboration with middle school mathematics teachers, we have become increasingly aware of the pervasive influence teachers' images have on how they implement innovative curricula. We have observed that these images manifest themselves in two sharply contrasting orientations toward mathematics teaching. We refer to these orientations as *calculational* and *conceptual*. To illustrate what we mean by a calculational and a conceptual orientation in teaching mathematics, we start with two vignettes. After the vignettes we discuss more generally what these orientations entail and their implications for classroom discourse and for students' learning. The article ends with a discussion of obstacles to adopting a conceptual orientation and a discussion of the implications these obstacles have for the professional preparation and development of mathematics teachers.

The vignettes depict two different teachers, each illustrative of an orientation. Our intent is to give the reader concrete examples of the kind of teaching—specifically the nature of the classroom discourse—that is characteristic of each orientation. The vignettes have been constructed from videotaped observations of actual lessons.

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Vignette 1

A seventh-grade teacher presents the following problem to his class:

> At some time in the future John will be 38 years old. At that time he will be 3 times as old as Sally. Sally is now 7 years old. How old is John now?

After allowing students time to think about the problem and to discuss their thinking with a classmate, the teacher calls for volunteers to explain how they thought about the problem in order to solve it. What follows are the responses offered by the students and the ensuing exchange between teacher and students:

T: Let's talk about this problem a bit. How is it that you thought about it?

S1: I divided 38 by 3 and I got 12 2/3. Then I subtracted 7 from 12 2/3 and got 5 2/3. *(Pause)* Then I subtracted that from 38 and got 32 1/3. *(Pause)* John is 32 1/3.

T: That's good! *(Pause)* Can you explain what you did in more detail? Why did you divide 38 by 3?

S1: *(Appearing puzzled by the question, S1 looks back at her work. She looks again at the original problem.)* Because I knew that John is older—3 times older.

T: Okay, and then what did you do?

S1: Then I subtracted 7 and got 5 2/3. *(Pause)* I took that away from 38, and that gave me 32 1/3.

T: Why did you take 5 2/3 away from 38?

S1: *(Pause)* To find out how old John is.

T: Okay, and you got 32 1/3 for John's age. That's good! *(Pause)* Yes, S2?

S2: Isn't the answer 21? *(Pause)* I multiplied 7 times 3 and I got 21.

T: Hum? Not quite. *(Pause)* How come you multiplied 7 times 3?

S2: It says that he is 3 times as old as Sally ... *(Pause)* and Sally is 7.

T: Oh, I see! *(Pause)* You're right, the problem says that John is 3 times as old as Sally, but that is when John is 38. That's at the time he is 38, which is at some time in the future. *(Pause)* Do you understand?

S2: Sort of.

T: Okay, how about you, S3? How did you think about it?

S3: I divided 38 by 3, and I subtracted that from 38. That's 25 and something. Then I added that to 7. I got the same thing as S1—32 something.

T: But you did it differently. Super! See? There are different ways to solve the same problem. *(Pause)* How about you, S4?

S4: I subtracted 7 from 38 and divided that by 3. *(Pause)* I got 10 something. Then I added that to 7. *(Pause)* I got that he is 17 and something.

T: Hum? That doesn't quite agree with the other answers, does it? I'm not sure I understand what you're doing. *(Pause)* Why did you subtract 7 from 38?

S4: *(Shrugging his shoulders)* I don't know.

T: S5?

S5: Dividing 38 by 3 can't be right! It doesn't come out even.

T: That doesn't matter, does it? We still get a number, don't we? *(Pause)* We get that Sally is 12 2/3. *(Pause)* Let's take a look at how to divide 38 by 3. Divide 3 into 38. *(Motioning with his hands in the air as if he were doing the long division on an imaginary chalkboard)* Three goes into 38 ten times, put up the 1, and 10 times 3 is 30. Thirty-eight minus 30 is 8. Three goes into 8 two times. Put up the 2, and 2 times 3 is 6. So 8 minus 6 is 2. The answer is 12 remainder 2, or 12 and 2/3. Okay? *(Pause)* Let's take a look at the two ways the problem was solved.

The teacher proceeds to demonstrate S1's and S3's solutions on the chalkboard and refers to both solutions as appropriate ways to think about the problem. This segment of the lesson ends and the class moves to work on another task.

Contrast the vignette given above with the one below, which illustrates an exchange of a very different nature between a teacher and his students. This exchange followed the presentation of the same problem as in vignette 1 to a group of seventh graders. Again, the exchange takes place after the students have had the opportunity to think about the problem and to discuss it with a classmate.

Vignette 2

T: Let's talk about this problem a bit. How is it that you thought about the information in it?

S1: Well, you gotta start by dividing 38 by 3. Then you take away ...

T: *(Interrupting)* Wait! Before going on to tell us about the calculations you did, explain to us why you did what you did. *(Pause)* What were you trying to find?

S1: Well, you know that John is 3 times as old as Sally, so you divide 38 by 3 to find out how old Sally is.

T: Do you all agree with S1's thinking?

Several students say, "Yes"; others nod their heads.

S2: That's not gonna tell you how old Sally is *now*. It'll tell you how old Sally is when John is 38.

T: Is that what you had in mind, S1?

S1: Yes.

T: *(To the rest of the class)* What does the 38 stand for?

S2: John's age in the future.

T: So 38 is not how old John is now. It's how old John will be in the future. *(Pause)* The problem says that when John gets to be 38 he will be 3 times as old as Sally. Does that mean "3 times as old as Sally is *now*" or "3 times as old as Sally will be when John is 38"?

Several students respond in unison, "When John is 38."

T: Are we all clear on S2's reasoning? *(Pause)*

S3: I started the same way, but I got stuck dividing. *(Pause)* Three doesn't go into 38 evenly. *(Pause)*

T: Don't worry about how to divide 38 by 3 now. That's not what's most important right now. What are you trying to find by dividing 38 by 3?

S3: Sally's age.

T: Sally's age when John is 38 years old. *(Pause)* You can use your calculator if you want to. *(Pause)* If you try it, you'll get 12.66 ... years. That's Sally's age in the future. *(Pause)* S4?

S4: Couldn't you just say John is 21? *(Pause)* Couldn't you just multiply 3 times 7?

T: What will that give you?

S4: Twenty-one!

T: Yes, I know that. But what would the 21 represent? What is it that's 21?

S4: That's how old John is now. Isn't that what we want to find?

S5: No! *(Pause)* I mean, yes! That's what we want to find, but that's not right!

T: What is it that is not right, S4? We do want to find out how old John is now, don't we?

S5: Right. But see, he's not 3 times older than Sally *now!* He'll be 3 times older than Sally when he is 38. So you can't multiply 7 by 3.

T: Let's think about that. If we know that John will be 3 times as old as Sally when he is 38, does that make him 3 times as old as Sally now? *(Pause)* S4, what do you think?

S4: I guess not. *(Pause)*

T: *(To S4)* Suppose you're now 12 and your younger sister or brother is 6 years old. That makes you twice as old as your younger sister. Will that also be true next year? *(Pause)* Next year you'll be 13 and she'll be 7. Will you still be twice as old as your sister?

S6: Actually, that'll happen only once and never again.

S4: I see it.

T: Okay, so how are we going to use the information that John will be 3 times as old as Sally when he gets to be 38? *(Pause)* Who can explain?

S1: You can divide 38 by 3 and get 12.66....

T: Remember to tell us what your numbers stand for. What does the 12.66 ... stand for?

S2: That's how old Sally will be.

T: When?

Several students respond, "When John is 38."

T: Okay, we know how old Sally will be when John is 38 years old. *(Pause)* She will be 12.66 ... years. We can say she'll be 12, because we usually don't say that we are 12.66 ... years old. We typically use whole numbers when we talk about our age. Okay?

S6: Okay, you can say that Sally will be 12. So if you subtract 7 from that you get 5. Then you take away 5 from 38 and you're done! John is 33.

T: Wait a minute! You're going too fast. I don't see how you know to do all that. Can you explain your reasoning?

S6: *(Patiently)* You know Sally will be 12 and something, and you know that she is 7 now. So that means that there are 5 years between now and then. Actually a little more than 5 years, but you said that was okay.

T: Yes, it's okay to say 5 years. So 5 years is how much time there is between now and the time in the future when John is 38?

S6: Yes, So if you take 5 away from 38, that's how old John is now.

T: Did everyone follow S6's reasoning? *(Pause)* Who can recap the solution we've just been through?

The teacher calls on two volunteers who, with some assistance from other classmates and the teacher, summarize the discussion.

T: Did anyone think about the problem differently? *(Pause)* S7?

S7: Well, sort of. I started out the same. I divided 38 by 3.

T: *(Interrupting)* To find what?

S7: Sally's age in the future.

T: Okay.

S7: I got that Sally will be 12 2/3 years old when John is 38. Then I subtracted to find the difference between their ages. *(Pause)* I got 25 1/3.

T: Twenty-five and one-third what?

S7: Twenty-five and one-third years. That's how much older John is. *(Pause)*

T: How much older than Sally?

S7: Yes. That's the difference between their ages.

T: Now or when John is 38?

S7: Actually, it doesn't matter. The difference between their ages will always be the same.

T: Okay, we can come back to that thought in a minute. *(Pause)* Go on.

S7: So to find out how old John is now.... See, you know Sally is now 7 and John is 25 1/3 years older than Sally. So add 25 1/3 to 7, and you get John's age. That's 32 1/3. *(Pause)* That's how I figured it.

T: Who agrees with S7's reasoning?

Several hands go up.

S8: I don't understand why she added 25 1/3.

S2: Because that's how much older John is than Sally.

S8: I still don't see why she added that to 7.

S2: If you know Sally is 7 and John is 25 1/3 years older than Sally, you add to get how old John is now.

S8: *(Puzzled)* But 25 1/3 is when John is 38 and Sally is 12 2/3.

S9: The difference between their ages is always the same—now and when John is 38.

T: Does that make sense to everyone? *(Pause)* Who can explain S7's solution method from the beginning? *(Pause)* Don't just tell me what operations she did. Remember, "to explain" means that you have to talk about her reasoning, not just the arithmetic she did.

The discussion continues. The teacher poses more questions aimed at focusing students' attention on the quantities and quantitative relationships in the problem. He probes for the reasoning underlying the students' arithmetic procedures. As the teacher elicits responses from the students, he sketches a diagram (fig. 8.1) to support the discussion of invariance of age differences and variance of age ratios.

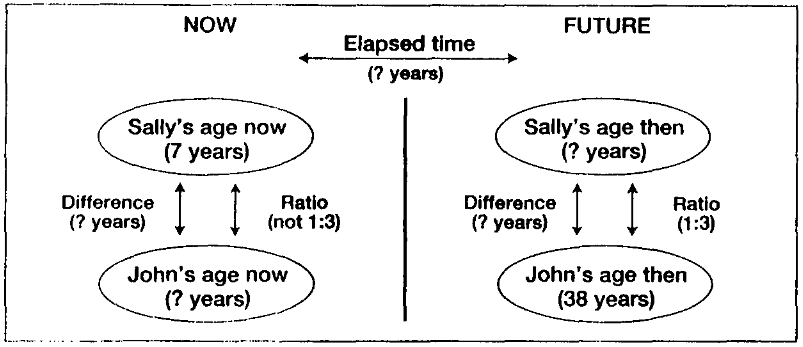

Fig. 8.1

THE VIGNETTES AND THE TEACHERS: SIMILARITIES AND DIFFERENCES

We constructed the vignettes from actual classroom observations to capture as concretely as possible what we have observed to be important differences in mathematics classroom discourse. Despite the obvious similarities, important substantive differences can be discerned in the two vignettes. Although both teachers opened their lessons with the same problem and with similar instructions, the ensuing discussions were quite different. They not only differed in superficial, albeit important, features, such as duration and number of students involved (the discussion of the problem in the first vignette was much briefer than in the second vignette, and vignette 1 overtly involved five students, whereas nine students contributed to the discussion in vignette 2), but also differed markedly in what was discussed and in the roles the teachers played.

In both vignettes the students initially offered sequences of arithmetic procedures as expressions of their thinking. However, in vignette 2 the students began to give explanations that were grounded in conceptions of the situation. In contrast, the explanations given by the students in vignette 1 remained strictly procedural; they were all statements of how they calculated John's age, and they all failed to address what the teacher ostensibly requested—an explanation of how they thought about the problem. The students in vignette 1 did not offer a justification for the chosen operations that was grounded in conceptions of the situation; when explaining, they did not connect their calculations to the ideas of time, duration, aging, or relationships among them. Their explanations were "calculational," which stand in sharp contrast to the conceptual explanations given in vignette 2.

Both teachers pressed their students to give rationales for their calculational solutions, but they did so differently and with quite different results. When compared with the explanations elicited by teacher 2, the explanations obtained by teacher 1 were shallow and incomplete (recall the student who

justified dividing 38 by 3 by saying that John is older than Sally). Teacher 1 was less persistent than teacher 2 in probing the students' thinking. He accepted solutions consisting of calculational sequences if they were correct by some criteria that he did not make explicit to the students. Teacher 2, in contrast, persistently probed students' thinking whenever their responses were cast in terms of numbers and operations, thus steering the discussion and focusing the students' attention on how they were conceiving the situation. His students were more inclined to comment on one another's contributions than were teacher 1's students.

Another important difference between the teachers was in their responses to the students' difficulties with dividing 38 by 3. Teacher 1 used the occasion as an opportunity to review the long-division algorithm; teacher 2 steered the students' attention away from the computational difficulty, downplaying its significance and redirecting their attention toward the quantitative relationship that suggested division.

The actions of the two teachers were driven by different images of their pedagogical tasks and of the goals they served. Teacher 1's image was that there was a problem to be solved. Teacher 2's image was of an occasion for students to reason and to reflect on their reasoning. Although it might be argued that for both teachers the general goal was the long-term development of the students' problem-solving skills, for teacher 2 that development clearly entailed getting the students skilled at reasoning. Furthermore, teacher 2 had an image of what is involved in becoming a skilled reasoner, which he obviously had translated into specific pedagogical practices. His actions appeared to be driven by the belief that not until students make their reasoning explicit to themselves can they reflect on it and represent it mathematically and that those representations empower their reasoning. The distinctions between these teachers' actions reside in their orientations toward mathematics and teaching mathematics. The teacher in vignette 1 exemplifies what we call a "calculational orientation." The teacher in vignette 2 exemplifies what we call a "conceptual orientation."

In the remainder of this paper we focus on these two orientations from a more theoretical perspective. First, we characterize the two orientations. Next, we address the consequences of each orientation for the teachers' instructional practices, the students' learning and beliefs, and the nature of the classroom discourse. We conclude with a discussion of the obstacles to adopting a conceptual orientation and some remarks about what might be involved in successfully adopting such an orientation.

TWO CONTRASTING ORIENTATIONS

We believe that the substantive differences in the way the teachers handled the curricular task in the vignettes are an expression of a fundamental difference in their orientations toward teaching mathematics. As mentioned above, we refer to these as "conceptual" and "calculational" orientations. Here is how we characterize them.

A teacher with a conceptual orientation is one whose actions are driven by—

- an image of a system of ideas and ways of thinking that she intends the students to develop;
- an image of how these ideas and ways of thinking can develop;
- ideas about features of materials, activities, and expositions and the students' engagement with them that can orient the students' attention in productive ways (a productive way of thinking generates a "method" that generalizes to other situations;
- an expectation and insistence that students be intellectually engaged in tasks and activities.

Conceptually oriented teachers often express the images described above in ways that focus students' attention away from the thoughtless application of procedures and toward a rich conception of situations, ideas, and relationships among ideas. These teachers strive for conceptual coherence, both in their pedagogical actions and in students' conceptions. As a result, conceptually oriented teachers tend to focus on aspects of situations that, when well understood, give meaning to numerical values and that suggest numerical operations (Thompson in press). Conceptually oriented teachers often ask questions that move students to view their arithmetic in a noncalculational context like the following:

- "(This number) is a number of what?"
- "To what does (this number) refer in the situation we're dealing with?"
- "What are you trying to find when you do this calculation (in the situation as you currently understand it)?"
- "What did this calculation give you (in regard to the situation as you currently understand it)?"

The actions of a teacher with a calculational orientation are driven by a fundamental image of mathematics as the application of calculations and procedures for deriving numerical results. This does not mean that such a teacher is focused only on computational procedures.[1] Rather, his view of mathematics is more inclusive but still one focused on procedures—computational or otherwise—for "getting answers."

These are some symptoms of a calculational orientation:

- A tendency to speak exclusively in the language of numbers and numerical operations
- A predisposition to cast solving a problem as producing a numerical solution

[1] This view we call a "computational orientation." A teacher with a computational orientation views mathematics as composed of computational procedures, and doing mathematics as computing in the absence of any reason for the computation aside from the context of having been asked to do so. A computational orientation implies a calculational orientation but a calculational orientation does not imply a computational orientation.

- An emphasis on identifying and performing procedures
- A tendency to do calculations whenever an occasion to calculate presents itself regardless of the overall context in which the occasion occurs
- A tendency to disregard the context in which the calculations might occur and how they might arise naturally from an understanding of the situation itself
- An inclination to remediate students' difficulties with calculational procedures independently of the context in which the difficulties manifest themselves
- A tendency to treat problem solving as flat; that is, nothing about problem solving is any more or less important than anything else, except that the answer is most important because getting the answer is the reason for solving the problem
- A narrow view of mathematical patterns as limited to finding patterns in numerical sequences and in the sameness of operations across problems, as opposed to finding patterns in reasoning in the solution of problems

Consequences of Calculational and Conceptual Orientations

Calculational and conceptual orientations can have different consequences for the interchanges that occur in classrooms (Wertsch and Toma in press). These consequences can be organized by the interplay between teachers and students according to which orientations each possesses and by the interplay among students possessing different orientations. We shall focus on the influence of teachers' orientations on classroom discourse because we believe that teachers set the tone for the kinds of discussions in which students engage, whether with the teacher or among themselves (Cohen 1990; Porter 1989; A. Thompson and P. Thompson 1994; P. Thompson and A. Thompson in press).

The teachers' goals and images described in the previous section account for many of the differences between the two vignettes. The first teacher's goal was for students to solve the problem and share their procedures; the second teacher's goal was to create an occasion for students to reason and to make their reasoning public. Subtle but important differences in the teachers' behaviors were an expression of their different goals.

In the previous section we described the teachers' pedagogical tasks. The teacher in vignette 1 expected his students to explain their procedures; the teacher in vignette 2 expected students to explain their reasoning. One manifestation of the teachers' goals is the type of questions they asked. For example, both teachers asked S1 why she had decided to divide 38 by 3. The second teacher also asked S1, "What were you trying to find when you divided 38 by 3?" By asking this question, the teacher oriented his students toward the

situation itself and their conception of it, which required the students to reflect on their understanding of the situation. An important feature of vignette 2 is that the teacher persisted in bringing students back to thinking about their conceptions of the situation. This orientation contrasts with orienting students to reflect on their calculations and with allowing students to remain oriented toward their calculations.

Students also have varying degrees of conceptual or calculational orientations to mathematics. Those who have adapted to calculationally oriented instruction will approach mathematical discussions with the expectation that they will be concerned with getting answers (Cobb, Yackel, and Wood 1989; Nicholls et al. 1990). Students who have come to view mathematics as "answer getting" not only will have difficulty focusing on their and others' reasoning but also may consider such a focus as being irrelevant to their images of what mathematics is about.

Conversely, students who have adopted a conceptual orientation will likely engage in longer, more meaningful discussions (Cobb, Wood, and Yackel 1991). Vignette 2 lasted longer and involved more students than vignette 1 because students had something to discuss. Students in vignette 1 did not sustain a substantive discussion because they had no way of knowing the sources of their classmates' procedures. Reasoning was not a subject to discuss. Students in vignette 2, through the support of their teacher, did discuss their reasoning and in doing so created an environment in which they felt free to share their understandings.

A calculationally oriented teacher may believe that explaining the calculations one has performed is tantamount to explaining one's reasoning (Cobb, Wood, and Yackel in press). We observed that the only students able to follow a calculational explanation are those who understood the problem in the first place and understood it in such a way that the proposed sequence of operations fits their conceptualization of the problem. To illustrate this observation, imagine four students, Alicia, Betty, Carl, and Don, all of whom solved the "Sally and John" problem incorrectly. Furthermore, imagine that their errors stemmed from different sources. Alicia missed the problem because she committed a calculational error, but her understanding of the problem was valid and she understood the problem in a way that fit the calculational explanation offered by S1. Betty missed the problem because of a calculational error; her understanding of the problem was valid, but her understanding of the problem did not fit the string of calculations offered by S1. Carl and Don missed the problem because they could not conceptualize it; Don possesses a calculational orientation and Carl possesses a conceptual orientation. The four students are listening to the discussion between S1 and T1 (the teacher in the first vignette):

S1: I divided 38 by 3 and I got 12 2/3. Then I subtracted 7 from 12 2/3 and got 5 2/3. Then I subtracted that from 38 and got 32 1/3. John is 32 1/3.

T1: That's good! *(Pause)* Can you explain what you did in more detail? Why did you divide 38 by 3?

S1: *(Appearing puzzled by the question, she looks back at her work. She looks again at the original problem.)* Because I knew that John is older—three times older.

T1: Okay, and then what did you do?

S1: Then I subtracted 7 and got 5 2/3. *(Pause)* I took that away from 38 and that gave me 32 1/3.

T1: Why did you take 5 2/3 away from 38?

S1: *(Pause)* To find out how old John is.

T1: Okay, and you got 32 1/3 for John's age. That's good!

For Alicia, who had made a calculational error but understood the problem in a way that fits S1's string of operations, this explanation validates her solution attempt, leaving her with the sense that she now understands what she had actually understood all along. Betty is convinced that she does not understand the problem at all—her initial answer was incorrect, and S1's string of operations do not fit with the way she conceived the problem. Don thinks he now understands, since he was able to follow all S1's calculations. Don's ability to perform all the calculations may even give him the confidence to explain S1's solution to Carl, who complains that he does not understand. However, Don's procedural explanation only leaves Carl even more frustrated, since he finds Don's explanation incomprehensible. These explanations do not tell Carl why the calculations were performed. In fact, with all the Dons in the class nodding as if they understand, Carl may judge that there is something wrong with his ability to understand mathematics, when in fact the only problem is that his expectations for understanding are greater than those of his peers. Over time, a conceptually oriented student such as Carl, sitting in a classroom dominated by calculationally oriented discourse, may conclude that mathematics is not supposed to make sense. Eventually, he may altogether stop trying to understand mathematics.

OBSTACLES AND IMPLICATIONS

It is evident to us that a conceptual orientation is by far the more enriching and the more productive for students and for teachers. But this orientation cannot be created easily, nor once created, can it be easily maintained (Romberg and Price 1981; von Glasersfeld 1988; Wood, Cobb, and Yackel 1991). To create a conceptual orientation, the teacher must reflect long and deeply on her goals for, and images of, mathematics and mathematics teaching. In our personal experience, there are periods of confusion about what we are trying to have our students understand, and teachers working with us have expressed the same feelings. When we move our focus of instruction to deep conceptualizations of situations, we also move away from the domains of discourse with which we feel most

comfortable—established methods for deriving numerical solutions. Instead, we move toward domains of discourse that emphasize "how you think about it"—domains few of us have explored and too few students have experienced.

One of the major obstacles to creating a conceptual orientation is teachers' lack of ideas about how to move pedagogically from holding conversations about "how you think about it" to the standard mathematics of the conventional curriculum. Teachers frequently ask us essentially this question: "After we've talked about understanding these situations, how do I introduce the standard procedures?" This question indicates to us a teacher who is grappling with a dilemma—how to reconcile an emphasis on students' reasoning with the traditional curriculum and pedagogy wherein symbols, methods, and procedures are introduced before students encounter any substantive applications.

A conceptual approach to teaching mathematics aims for students to solve problems by working from a deep understanding of them. But working from an understanding means that they work from *their* understandings. A primary aim of conceptually oriented teaching is that students come to conceive a conceptual domain by developing methods for solving problems in it. Part of students' developing stable, general methods is that they deal with the matter of expressing those methods in notation. Once students have developed conceptual methods and have reflected those methods in notation, they can then appreciate that conventional methods are but one way to solve problems in a conceptual domain. It is important that students also appreciate that the most powerful approach to solving problems is to understand them deeply and proceed from the basis of understanding and that a weak approach is to search one's memory for the "right" procedure. A teacher's dilemma regarding when to introduce conventional procedures is eventually resolved when this teacher realizes that no reconciliation is possible; the traditional curriculum turns the construction of mathematical meaning upside down. The resolution of the dilemma comes from the teacher's creation of a new philosophy—a philosophy of what he or she is trying to attain that permeates his or her instructional goals and actions (Ball 1993).

Once a teacher makes a commitment to treat mathematics conceptually, she loses the support structures on which she has come to rely, such as textbooks and repertories of stable practices. This loss is very threatening and thus is a major obstacle to change. Old habits die hard and new practices evolve slowly. For most teachers who lack the time and energy to rethink their curriculum and pedagogy, the thought of giving up conventional materials can be very unsettling. Our research suggests that having a repository of rich problems is enough to begin moving away from the textbook. Our research also suggests that such a repository is not sufficient to ensure success; a conceptual understanding of the subject matter the problems address is also necessary for teachers to feel that they have a sense of direction and to be able to respond to students' difficulties.

To teach mathematics conceptually, it is not sufficient to know how to solve the problem with which the students may be grappling, nor even to know several methods of solution (McDiarmid, Ball, and Anderson 1989). To teach conceptually requires a deep conceptualization of the situation. This, in turn, requires that the teacher think beyond what is necessary to simply find ways of dealing mathematically with the situation. Furthermore, to be able to orient students' thinking in productive ways, it is extremely helpful to have an image of students' thinking as they develop these ideas. Any teacher can begin building this image by encouraging students to reason and express themselves accordingly and by listening to their reasoning, respecting it, and asking the other students to do likewise.

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